A General Theory of Osculation Algorithms for Conformal Mapping

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To Alexander Ostrowski on his 90th birthday

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ABSTRACT

Let $\mathfrak{D}$ be the family of simply connected regions D such that $0 \in D \subset E$, the unit disk, and let $\mathfrak{D}(\rho)$ be the set of all $D \in \mathfrak{D}$ such that $\text{dist}(\partial D, 0) = \rho$. An *osculation family* assigns to every $D \in \mathfrak{D}$ a family $\mathfrak{F}(D)$ of conformal maps h of D such that: (1) $h(D) \in \mathfrak{D}$, $h(0) = 0$, $h'(0) > 0$; (2) the inverse map $h^{[-1]}$ can be extended to an analytic function on E with values in E; (3) $\gamma(\rho) := \inf_{D \in \mathfrak{D}(\rho)} \inf_{h \in \mathfrak{F}(D)} h'(0)$ satisfies $\inf_{\rho \in I} \gamma(\rho) > 1$ for every $I := (0, \beta]$, where $\beta < 1$. (Example: the Koebe family, but there are many others.) We show that $\gamma(\rho) \leqslant (1+\rho)^2/4\rho$ for any osculation family, the bound being best possible for all $\rho \in (0,1]$. An *osculation algorithm* constructs the Riemann map of a $D \in \mathfrak{D}$ onto E as $\lim h_n \circ h_{n-1} \circ \cdots \circ h_1$, where $D_1 := D$ and $D_{n+1} := h_n(D_n)$, h_n being any map in $\mathfrak{F}(D_n)$. We prove convergence and, if $D_n \in \mathfrak{D}(\rho_n)$ and $\gamma(\rho) \geqslant 1 + \alpha(1-\rho)^2$, show that $1 - \rho_n \leqslant 8\alpha^{-1}\rho_1^{-2}n^{-1}$, thereby generalizing and improving a result of Ostrowski (1929). If the unique solution $\phi = \phi(\rho)$ of $\phi = \rho\exp\{[(1-\gamma\phi)/(1+\gamma\phi)]\text{Log}\,\gamma\}[\gamma := \gamma(\rho)]$ is increasing, we also show that $\rho_n \geqslant \rho_n'$, where $\rho_1' := \rho_1$, $\rho_{n+1}' := \phi(\rho_n')$, thus explaining the experimentally observed rapid convergence of the osculation algorithm during the first few iteration steps.

1. INTRODUCTION

The Riemann mapping theorem asserts the unique existence of an analytic function f that maps a given simply connected region $D \neq \mathbb{C}$ onto the unit disk E while sending a preassigned point $a \in D$ into 0 and satisfying $f'(a) > 0$. Ostrowski [16] noted as early as 1929 that the proof of the mapping theorem which is still current today (see e.g. [4, p. 6], [1, p. 172], [5, p. 156], [12, p. 152]) is nonconstructive, because it requires the extraction of convergent subsequences. Many numerical methods for the effective construction (or approximation) of the mapping function f are known (see the surveys by

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Gaier [6] or Opfer [15]). Among all these methods, the *Schmiegungsverfahren* ("osculation method") of Koebe [14] enjoys a special position not only in view of its conceptual simplicity, but also because it is universally applicable, since it requires no hypotheses concerning the boundary ∂D of D.

In the Schmiegungsverfahren it is assumed that $D \subset E$, $a = 0$, which can always be achieved by a simple preliminary transformation. Letting $D_1 := D$, a sequence of simply connected regions $\{D_n\}$ is constructed by means of a simple elementary map (Koebe's *Schmiegungsfunktion*). The sequence $\{D_n\}$ is such that $D_n \subset E$ and $0 \in D_n$ for all n, and that $\rho_n := \operatorname{dist}(\partial D_n, 0)$ increases monotonically to 1. The boundaries ∂D_n of D_n thus "osculate" more and more to the unit circle, and the maps from D_1 to D_n, which are compositions of elementary maps, approximate the required mapping function f.

Concerning the speed of convergence of the *Schmiegungsverfahren*, Ostrowski [16] proved that $1 - \rho_n = O(1/n)$. Although an asymptotically bad *estimate* of the speed of convergence does not of course imply that the *actual speed* of convergence is slow, Ostrowski's result seems to have been taken as an indication that the numerical performance of Koebe's method would be poor. Modifications of the method, aimed at improving the convergence in special cases, were suggested by Heinhold [8,9] and Albrecht [3]. Hübner [10, 11] applied the idea of osculation to obtain mappings onto the upper half plane and implemented his method in simple cases. Little comprehensive experimentation with osculation methods, however, has been carried out until recently. By his extensive experiments, Grassmann [7] obtained a more discriminating view of the method. Grassmann's main conclusions, confirming in part those reached by Hübner in simple cases, were as follows: (i) Asymptotically, the convergence of the *Schmiegungsverfahren* is indeed very poor. To speak concretely, once ρ_n reaches a value such as 0.99, it takes innumerable iterations to make further progress. (ii) Initially, however, the convergence of the *Schmiegungsverfahren* is much better. To get to $\rho_n = 0.99$ takes fewer iterations than might be expected from Ostrowski's estimate. (iii) The initial convergence of the method is capable of significant further improvement if Koebe's *Schmiegungsfunktion* is replaced by other elementary mapping functions which, although less universally applicable, are better adapted to a special geometric situation at hand.

The present paper aims at providing some theoretical support for Grassmann's experimental conclusions. We define a general osculation algorithm which contains Heinhold's, Albrecht's, and Grassmann's modifications as special cases, and we prove its convergence (Theorem 5). For the general osculation algorithm we then establish in Theorem 6 an upper bound for $1 - \rho_n$ which, although again $O(1/n)$, improves Ostrowski's bound for the Koebe algorithm. We next improve and generalize Ostrowski's results on a "perturbed" Koebe algorithm. We finally establish some bounds for ρ_n

(Theorems 7 and 8) which, although asymptotically worse than the $O(1/n)$ bound, furnish better results initially.

The concepts introduced in this paper are useful also for the discussion of osculation methods for the mapping of doubly connected regions; see Hoidn [18].

2. OSCULATION FAMILIES

Let $\mathfrak{D}$ be the set of all simply connected regions $D \subset E$ such that $0 \in D$, and for each $\rho \in (0,1]$, let $\mathfrak{D}(\rho)$ denote the subset of $\mathfrak{D}$ consisting of all D such that $\min\{|z|: z \in \partial D\} = \rho$.

We now assume that for each $D \in \mathfrak{D}$, $D \neq E$, there is defined a nonempty collection $\mathfrak{F}(D)$ of conformal maps h of D with the following properties:

(A) For each $h \in \mathfrak{F}(D)$, we have $h(D) \in \mathfrak{D}$ and

$$h(0) = 0, \qquad h'(0) > 0; \tag{1}$$

(B) the inverse map $h^{[-1]}$ can be extended analytically (but not necessarily as a 1-1 function) into E in such a way that $h^{[-1]}(E) \subset E$;

(C) if

$$\gamma(\rho) := \inf_{D \in \mathfrak{D}(\rho)} \inf_{h \in \mathfrak{F}(D)} h'(0), \qquad 0 < \rho < 1, \tag{2}$$

then for any interval $I := (0, \beta]$ where $\beta < 1$,

$$\inf_{\rho \in I} \gamma(\rho) > 1. \tag{3}$$

A family $\mathfrak{F} = \bigcup_{D \in \mathfrak{D}} \mathfrak{F}(D)$ is called an *osculation family*, and the function $\gamma(\rho)$ defined by (2) is the *dilatation measure*[1] of $\mathfrak{F}$.

EXAMPLE 1. A simple osculation family is obtained by letting $\mathfrak{F}(D)$ consist of the Koebe function for each $D \in \mathfrak{D}$, $D \neq E$. This is the special case $m = 2$ of the osculation family considered in Example 2.

EXAMPLE 2. Let $D \in \mathfrak{D}(\rho)$, $0 < \rho < 1$, and let z^* be a point of ∂D closest to the origin. By a preliminary rotation which has to be undone afterwards,

[1] Not to be confused with the *dilatation quotient* of a quasi-conformal mapping.

we may assume that z^* lies on the axis of negative reals, $z^* = -\rho$. By

$$z \to z_1 := \frac{\rho + z}{1 + \rho z}$$

D is mapped onto a simply connected region $D_1 \subset E$ which contains $z_1 = \rho$ and which has 0 as a boundary point. Thus for any integer $m \geqslant 2$, the map

$$z_1 \to z_2 := z_1^{1/m}$$

can be defined as a one-to-one analytic function in D_1 whose range D_2 is made unique by requiring that $\rho \to \rho^{1/m} > 0$. Finally,

$$z_2 \to w := \frac{z_2 - \rho^{1/m}}{1 - \rho^{1/m} z_2}$$

maps D_2 onto a region $D_3 \in \mathfrak{D}$. The composition of the foregoing three maps,

$$z \to w = h(z) := \frac{\left[\dfrac{z+\rho}{1+\rho z}\right]^{1/m} - \rho^{1/m}}{1 - \rho^{1/m}\left[\dfrac{z+\rho}{1+\rho z}\right]^{1/m}},$$

clearly satisfies (A). The inverse map is calculated to be

$$z = h^{[-1]}(w) = \frac{\left[\dfrac{w+\rho^{1/m}}{1+\rho^{1/m} w}\right]^{m} - \rho}{1 - \rho\left[\dfrac{w+\rho^{1/m}}{1+\rho^{1/m} w}\right]^{m}}, \tag{4}$$

and this evidently satisfies (B). Since

$$h'(0) = \frac{1}{m}\,\frac{\rho^{-1} - \rho}{\rho^{-1/m} - \rho^{1/m}}$$

independently of $D \in \mathfrak{D}(\rho)$, it follows that

$$\gamma(\rho) = \frac{1}{m}\,\frac{\rho^{-1} - \rho}{\rho^{-1/m} - \rho^{1/m}}.$$

For $m = 2$,

$$\gamma(\rho) = \frac{1+\rho}{2\sqrt{\rho}} > 1,$$

and because $m(\rho^{-1/m} - \rho^{1/m})$ decreases to its limit $2\,\mathrm{Log}(1/\rho)$ as $m \to \infty$, it follows that (C) is satisfied for every integer $m \geqslant 2$. Thus by letting $\mathcal{F}(D)$ consist of some or all of the above functions h, another osculation family is obtained.

EXAMPLE 3. The limit as $m \to \infty$ of the function h of Example 2 is

$$h(z) = \frac{\mathrm{Log}\,\rho - \log \dfrac{z+\rho}{1+\rho z}}{\mathrm{Log}\,\rho + \log \dfrac{z+\rho}{1+\rho z}},$$

where $\log[(z+\rho)/(1+\rho z)]$ is analytic in D and has the real value $\mathrm{Log}\,\rho$ for $z = 0$. This again satisfies (A). The inverse function is

$$z = h^{[-1]}(w) = \frac{\rho^{(1-w)/(1+w)} - \rho}{1 - \rho \cdot \rho^{(1-w)/(1+w)}},$$

and this clearly satisfies (B). Computation shows

$$\gamma(\rho) = \inf_{D \in \mathfrak{D}(\rho)} h'(0) = \frac{\rho^{-1} - \rho}{2\,\mathrm{Log}\,\rho^{-1}},$$

which is > 1 for $0 < \rho < 1$. We call the osculation family thus defined the Ahlfors family (see [1, p. 173]; not in later editions).

3. AUGMENTATION OF AN OSCULATION FAMILY

Here we show that it is sometimes possible to add functions to a set $\mathcal{F}(D)$ without changing the dilatation measure $\gamma(\rho)$.

THEOREM 1. *Let $D_1 \in \mathfrak{D}(\rho)$, where $0 < \rho < 1$, and let f be the function, normalized by* (1), *mapping D_1 onto E. Then f may be added to any set $\mathcal{F}(D)$ where $D \in \mathfrak{D}(\rho)$, $D \subset D_1$, without changing $\gamma(\rho)$.*

Proof. Clearly, $f(D) \in \mathfrak{D}$, and the properties (A) and (B) are obvious for the restriction of f to any $D \subset D_1$. To establish (C), let h_1 be any function in $\mathfrak{F}(D_1)$. Then $h_1 \circ f^{[-1]}$ is analytic in E, zero at 0, and its values are in E. Thus by the Schwarz lemma,

$$\left|h_1 \circ f^{[-1]}(w)\right| \leqslant |w|, \qquad w \in E,$$

$$\left|h_1'(0) f^{[-1]\prime}(0)\right| \leqslant 1;$$

hence $h_1'(0) \leqslant f'(0)$, and since $h_1'(0) \geqslant \gamma(\rho)$, the conclusion follows. ∎

Example 4 (The crescent map). Let $D \in \mathfrak{D}(\rho)$ be contained in the set $E \setminus K \in \mathfrak{D}(\rho)$, where K is a disk that intersects the boundary of E. Then $\mathfrak{F}(D)$ may be augmented by the function f_1 (sometimes called the crescent map) mapping D_1 onto E. This map is elementary; see [7] for a computer-oriented implementation.

Example 5 (The slit map). This maps the disk cut along the straight line segment from -1 to $-\rho$ onto the full disk; the map may thus be added to any $\mathfrak{F}(D)$ where $D \in \mathfrak{D}(\rho)$ does not contain that line segment. The explicit formula is

$$f_1(z) = \frac{s(z) - 1 + z}{s(z) + 1 - z}, \tag{5}$$

where

$$s(z) := \sqrt{(1 + \rho z)(1 + \rho^{-1} z)}, \qquad s(0) = 1.$$

We note that

$$f'(0) = \tfrac{1}{4}(\rho + 2 + \rho^{-1}) = \left(\frac{1+\rho}{2\sqrt{\rho}}\right)^2. \tag{6}$$

4. AN UPPER BOUND FOR THE DILATATION FUNCTION

Let D_ρ denote the region considered in Example 5 (E with segment $[-1, -\rho]$ removed), and let $\mathfrak{F}$ be any osculation family. Since, by Theorem 1, the slit map may be added to $\mathfrak{F}(D_\rho)$ without decreasing the dilatation measure $\gamma(\rho)$ of $\mathfrak{F}$, there follows in view of (6):

THEOREM 2. *For any osculation family $\mathfrak{F}$, the dilatation measure $\gamma(\rho)$ satisfies*

$$\gamma(\rho) \leqslant \tfrac{1}{4}(\rho + 2 + \rho^{-1}). \tag{7}$$

If $\rho = 1 - \varepsilon$, (7) means the same as

$$\gamma(1-\varepsilon) \leqslant 1 + \tfrac{1}{4}\varepsilon^2(1-\varepsilon)^{-1},$$

from which we conclude:

COROLLARY 3. *No dilatation measure can satisfy an inequality*

$$\gamma(1-\varepsilon) \geqslant 1 + \alpha\varepsilon^2$$

for all $\varepsilon \in (0,1)$ with a constant $\alpha > \frac{1}{4}$.

The question arises whether the inequality (7) is best possible, or in other words, whether there exists an osculation family such that its dilatation measure satisfies

$$\gamma(\rho) = \tfrac{1}{4}(\rho + 2 + \rho^{-1}). \tag{8}$$

This is answered by the following example.

EXAMPLE 6 (The Riemann family). Here each $\mathfrak{F}(D)$ consists of a single element only, namely of the normalized Riemann map of D onto E. It is clear that the Riemann family satisfies (A) and (B). To compute its dilatation measure, let $D \in \mathfrak{D}(\rho)$. Without loss of generality we assume that $z = -\rho$ is a point of ∂D closest to 0. Let f be the normalized Riemann map of D, and let k denote the function

$$k(z) := \frac{z}{(1-z)^2},$$

familiar from the theory of *schlicht* functions. It is known that k defines a one-to-one map of E. Consider

$$g(w) := f'(0) \cdot k \circ f^{[-1]}(w).$$

This defines a one-to-one map of E satisfying $g'(0) = 1$. By the Koebe–

Bieberbach $\frac{1}{4}$ theorem (see [2, p. 85, Theorem 16.1]), the image of E under g contains the disk $|z| < \frac{1}{4}$. Thus if $\{w_n\}$ is a sequence of points in E such that $z_n := f^{[-1]}(w_n) \to -\rho$, then

$$f'(0)|k(-\rho)| = \lim_{n\to\infty} f'(0)|k(z_n)|$$
$$= \lim_{n\to\infty} |g(w_n)| \geq \tfrac{1}{4}.$$

In view of

$$k(-\rho) = -\frac{\rho}{(1+\rho)^2}$$

this implies

$$f'(0) \geq \frac{(1+\rho)^2}{4\rho}.$$

Equality is attained for the mapping function of Example 5. Thus for the Riemann family,

$$\gamma(\rho) = \frac{(1+\rho)^2}{4\rho}.$$

5. THE OSCULATION ALGORITHM

Let $\mathcal{F}$ be an osculation family, and let $h \in \mathcal{F}(D)$, where $D \in \mathfrak{D}(\rho)$, $\rho < 1$. By (A) and (B), the function $h^{[-1]}$ satisfies the hypotheses of the Schwarz lemma. By (C), $h^{[-1]\prime}(0) \leq \gamma(\rho)^{-1} < 1$; thus $h^{[-1]}$ is not a rotation. There follows

$$|h^{[-1]}(w)| < |w|, \qquad w \in E, \quad w \neq 0. \tag{9}$$

Letting $w = h(z)$, where $z \in D$, we obtain the first assertion of

LEMMA 4. *For $z \in D$, $z \neq 0$,*

$$|h(z)| > |z|. \tag{10}$$

Furthermore

$$h(D) \in \mathfrak{D}(\rho'), \tag{11}$$

where $\rho' > \rho$.

Proof of (11). This does not directly follow from (10), because h need not be defined on ∂D, and by taking limits the inequality might degenerate into an equality. However, let w^* be a boundary point of $h(D)$ closest to 0, so that $|w^*| = \rho'$. If $\rho' = 1$, then (11) is clear. If $w^* \in E$, let $\{w_k\}$ be a sequence of points in $h(D)$ that converges to w^*. Let $z_k := h^{[-1]}(w_k)$. Because $h^{[-1]}$ is analytic at w^*, $\{z_k\}$ converges, and the limit $z^* := h^{[-1]}(w^*)$ necessarily is a boundary point of D. By (9) there follows

$$\rho \leqslant |z^*| < |w^*| = \rho',$$

proving (11). ∎

Given an osculation family, the following *osculation algorithm* for mapping a given region $D \in \mathfrak{D}$ onto E thus makes sense. Let $D_1 := D \in \mathfrak{D}(\rho_1)$. Select $h_1 \in \mathfrak{F}(D_1)$, and put $f_1 := h_1$. By virtue of Lemma 4,

$$D_2 := h_1(D_1) = f_1(D_1) \in \mathfrak{D}(\rho_2),$$

where $\rho_2 > \rho_1$. Generally, having constructed f_{n-1} and having obtained the region $D_n := f_{n-1}(D_1)$, we select $h_n \in \mathfrak{F}(D_n)$ and put $f_n := h_n \circ f_{n-1}$. Again by (11),

$$D_{n+1} := h_n(D_n) = f_n(D_1) \in \mathfrak{D}(\rho_{n+1}),$$

where $\rho_{n+1} > \rho_n$.

The functions f_n are also given by

$$f_n = h_n \circ h_{n-1} \circ \cdots \circ h_1, \qquad n = 1, 2, \ldots.$$

By virtue of (A), they satisfy

$$f_n(0) = 0. \tag{12}$$

Furthermore, by using the chain rule,

$$f_n'(0) = h_n'(0) h_{n-1}'(0) \cdots h_1'(0) > 0. \tag{13}$$

All f_n thus are members of the normal family considered in the usual proof of the Riemann mapping theorem, and it is clear that a suitable subsequence converges. But in fact we have

THEOREM 5. *The whole sequence* $\{f_n\}$ *(and not merely a selected subsequence) converges. The convergence is locally uniform in* D_1, *and the limit function* f *maps* D_1 *conformally onto* E *in such a way that* $f(0)=0, f'(0)>0$.

Proof. Each function $z \to f_n(\rho_1 z)$ maps E into E and 0 onto 0. Therefore by the Schwarz lemma,

$$f_n'(0) < \rho_1^{-1}, \qquad n=1,2,\ldots.$$

By (13), using (C), there follows

$$\gamma(\rho_1)\gamma(\rho_2)\cdots\gamma(\rho_n) < \rho_1^{-1}, \qquad n=1,2,\ldots. \tag{14}$$

Since all $\gamma(\rho_k)>1$, there necessarily holds

$$\lim_{k\to\infty} \gamma(\rho_k)=1,$$

which by (C) implies

$$\lim_{k\to\infty} \rho_k = 1.$$

The regions D_k thus tend to E. To prove the uniform convergence of the sequence $\{f_n\}$ on compact subsets $\hat{D}_1 \subset D_1$, we use a version of Harnack's theorem which asserts the existence of a constant μ (depending only on $\hat{D}_1$ and on D_1) such that for any function p that is analytic in D_1 and has a positive real part, there holds

$$|p(z)| \leqslant \mu |p(0)| \qquad \text{for all} \quad z \in \hat{D}_1.$$

To obtain a function with a positive real part we note that for $n>m$

$$\frac{f_n(z)}{f_m(z)} = \frac{1}{f_m(z)} h_n \circ h_{n-1} \circ \cdots \circ h_{m+1}(f_m(z))$$

is analytic and $\neq 0$ in D_1, because both denominator and numerator vanish only at $z=0$, where they both have a simple zero. Moreover by Lemma 4,

$$\left|\frac{f_n(z)}{f_m(z)}\right| > 1, \qquad z \in D_1. \tag{15}$$

Because D_1 is simply connected, it is possible to define an analytic logarithm,

$$p(z) = \log \frac{f_n(z)}{f_m(z)},$$

that is made unique by requiring that

$$p(0) = \operatorname{Log} \frac{f_n'(0)}{f_m'(0)}$$

is real. In view of (15), $\operatorname{Re} p(z) > 0$. The function $f_n \circ f_m^{[-1]}$ maps a disk of radius ρ_{m+1} into E and leaves the origin unchanged; hence by the Schwarz lemma

$$\left(f_n \circ f_m^{[-1]}\right)'(0) = \frac{f_n'(0)}{f_m'(0)} < \rho_{m+1}^{-1} < \rho_m^{-1}.$$

By Harnack's theorem we thus conclude

$$\left|\log \frac{f_n(z)}{f_m(z)}\right| \leqslant \mu \operatorname{Log} \rho_m^{-1}, \qquad z \in \hat{D}_1.$$

To derive from this an estimate for $f_n - f_m$, we use the elementary fact that for arbitrary $\eta > 0$ and arbitrary complex u such that $|u| \leqslant \eta$,

$$|e^u - 1| \leqslant \eta e^\eta.$$

(This follows in view of

$$|e^u - 1| = \left|\int_0^u e^t \, dt\right| \leqslant \int_0^u |e^t| |dt| \leqslant e^{|u|}|u|.)$$

Using this with $u := p(z)$, $\eta := \mu \operatorname{Log} \rho_m^{-1}$, we get

$$\left|\frac{f_n(z)}{f_m(z)} - 1\right| \leqslant \mu \rho_m^{-\mu} \operatorname{Log} \rho_m^{-1}, \qquad z \in \hat{D}_1,$$

which, on multiplying by $f_m(z)$, observing that $|f_m(z)| \leqslant 1$, and using the elementary inequality

$$\operatorname{Log}\frac{1}{\xi} \leqslant \frac{1}{\xi} - 1,$$

yields

$$|f_n(z) - f_m(z)| \leqslant \mu \rho_m^{-\mu-1}(1-\rho_m). \tag{16}$$

Since $\rho_m \to 1$, the expression on the right can be made as small as we please by choosing m large enough. The sequence $\{f_n\}$ for $z \in \hat{D}_1$ thus satisfies the Cauchy criterion, and the limit function f exists and is analytic. ∎

By the theorem of Hurwitz we see, as in the usual proof of the Riemann mapping theorem, that f is one-to-one. That f assumes every value in E follows from the fact that the domain of values of f contains every disk $|z| < \rho_m$, and thus by virtue of $\rho_m \to 1$ is the unit disk. The foregoing thus represents a constructive proof of the Riemann mapping theorem.

6. AN ESTIMATE FOR THE SPEED OF CONVERGENCE

A first indication of the speed of convergence of the osculation algorithm may be gleaned from the convergence of the product (13). Since the partial products are bounded by a quantity that does not depend on the chosen osculation family, it follows that the convergence will be fastest if the h_n are chosen to make $h_n'(0)$ as large as possible. This explains the speedup of convergence, observed experimentally by Grassmann [7], if maps such as the crescent and the slit map (see Examples 4 and 5) are used whenever applicable. Our observation would also predict that the Ahlfors family of Example 3 will produce faster convergence than the Koebe family. This as yet remains to be verified.

More precise statements concerning the speed of convergence are possible if the osculation family $\mathfrak{F}$ is such that its dilatation measure $\gamma(\rho)$ decreases for increasing ρ. (All our examples are of this kind.) For convenience we set

$$\rho = 1 - \varepsilon, \qquad \rho_n = 1 - \varepsilon_n, \quad n = 1, 2, \ldots.$$

We then have

THEOREM 6. *Let the dilatation measure be decreasing, and let there exist* $\alpha > 0$ *such that*

$$\gamma(1-\varepsilon) \geqslant 1 + \alpha\varepsilon^2, \qquad 0 < \varepsilon < 1. \tag{17}$$

(*We know from Corollary* 3 *that necessarily* $\alpha \leqslant \frac{1}{4}$.) *Then*

$$\varepsilon_n = \frac{8}{\alpha \rho_1^2} \frac{1}{n}, \qquad n = 1, 2, \ldots . \tag{18}$$

Proof. Because any D_n could serve as D_1, (14) implies

$$\gamma(\rho_n)\gamma(\rho_{n+1}) \cdots \gamma(\rho_{n+m}) < \rho_n^{-1}, \qquad n = 1, 2, \ldots, \quad m = 0, 1, \ldots .$$

Because $\{\rho_k\}$ is increasing, the sequence $\{\gamma(\rho_k)\}$ decreases, and there follows

$$[\gamma(\rho_{2n})]^{n+1} < \rho_n^{-1},$$

or

$$\gamma(\rho_{2n}) < \rho_n^{-1/(n+1)} < \rho_n^{-1/n}.$$

By the hypotheses,

$$1 + \alpha \varepsilon_{2n}^2 \leqslant \gamma(1 - \varepsilon_{2n}) = \gamma(\rho_{2n});$$

on the other hand, using Taylor's formula for $(1 - x)^{-\beta}$,

$$\rho_n^{-1/n} = (1 - \varepsilon_n)^{-1/n} \leqslant 1 + \frac{1}{n} \rho_n^{-2} \varepsilon_n \leqslant 1 + \frac{1}{n} \rho_1^{-2} \varepsilon_n .$$

There follows for $n = 1, 2, \ldots$

$$\alpha \varepsilon_{2n}^2 \leqslant \frac{1}{n} \rho_1^{-2} \varepsilon_n ,$$

$$\alpha \rho_1^2 n \varepsilon_{2n}^2 \leqslant \varepsilon_n ,$$

$$\tfrac{1}{4} \alpha \rho_1^2 (2n)^2 \varepsilon_{2n}^2 \leqslant n \varepsilon_n ,$$

$$(2n)^2 \left[\frac{\alpha \rho_1^2}{4} \right]^2 \varepsilon_{2n}^2 \leqslant n \frac{\alpha \rho_1^2}{4} \varepsilon_n .$$

Letting

$$\xi(n) := \tfrac{1}{4} \alpha \rho_1^2 n \varepsilon_n ,$$

we thus see that $[\xi(2n)]^2 \leqslant \xi(n)$; hence

$$\left[\xi(2^k)\right]^{2^k} \leqslant \xi(1) = \tfrac{1}{4}\alpha\rho_1^2(1-\rho_1), \qquad k = 0,1,\ldots .$$

The last expression is <1 in view of $\alpha \leqslant \frac{1}{4}$, and we therefore see that

$$\xi(2^k) < 1, \qquad k = 0,1,2,\ldots,$$

or equivalently,

$$\varepsilon_n < \frac{4}{\alpha\rho_1^2}\frac{1}{n}$$

whenever n is a power of 2. Since ε_n decreases, there holds for all $n = 1,2,\ldots$

$$\varepsilon_n < \frac{4}{\alpha\rho_1^2}\cdot\frac{1}{n'},$$

where n' is the greatest power of 2 not exceeding n. In view of $n' > n/2$, the result (18) follows. ∎

EXAMPLE 7. For the Koebe family (case $m = 2$ of Example 2),

$$\gamma(\rho) = \tfrac{1}{2}\left(\rho^{1/2} + \rho^{-1/2}\right).$$

By Theorem 2, we have $\gamma'(1) = 0$, and computation shows that

$$\gamma''(\rho) = \tfrac{1}{8}\rho^{-3/2}\left(3\rho^{-1} - 1\right) \geqslant \tfrac{1}{4},$$

$0 < \rho \leqslant 1$. Thus by Taylor's theorem

$$\gamma(1-\varepsilon) = 1 + \tfrac{1}{2}\gamma''(1-\theta\varepsilon)\varepsilon^2$$

for some $\theta \in (0,1)$. Hence

$$\gamma(1-\varepsilon) \geqslant 1 + \tfrac{1}{8}\varepsilon^2,$$

and the hypothesis of Theorem 6 is seen to be satisfied for $\alpha = \frac{1}{8}$. It thus follows that

$$\varepsilon_n \leqslant \frac{64}{\rho_1^2}\frac{1}{n}, \qquad n = 1,2,\ldots .$$

Ostrowski [16] proved (for the Koebe family only)

$$\varepsilon_n \leqslant 128 \frac{\operatorname{Log} \rho_1^{-1}}{1-\rho_1} \frac{1}{n},$$

which for $\rho_1 \to 1$ exceeds our estimate by the factor 2.

In a similar manner it can be shown that if the osculation family of Example 2 is used with arbitrary m,

$$\alpha = \tfrac{1}{6}(1 - m^{-2}),$$

and if the Ahlfors family (Example 3) is used,

$$\alpha = \tfrac{1}{6}.$$

Thus the error estimate of Theorem 6 becomes increasingly favorable with larger values of m, attaining the most favorable value for the Ahlfors family.

7. STABILITY OF THE OSCULATION ALGORITHM

Ostrowski [16] raised the question whether the osculation algorithm still converges, and if so at what speed, if the Koebe map is applied not with regard to some point z^* of ∂D closest to 0, but with regard to some point $z^* \in \partial D$ such that $\rho^* := |z^*| > \rho$ where $D \in \mathfrak{D}(\rho)$. From our more general point of view this simply amounts to augmenting the subfamily $\mathfrak{F}(D)$ of the osculation family $\mathfrak{F}$ by the restriction to D of a function $h^* \in \mathfrak{F}(D^*)$, where $D \subset D^* \in \mathfrak{D}(\rho^*)$. This means that the dilatation measure $\gamma(\rho)$ must be replaced by $\min\{\gamma(\rho), \gamma(\rho^*)\}$, which if γ is decreasing (as it normally will be) of course equals $\gamma(\rho^*)$. It follows that if the choice of ρ^* is such that $\rho^* \to 1$ only if $\rho \to 1$, then the general convergence theorem (Theorem 5) still holds. An $O(1/n)$ estimate of the form (18) still holds if the choice of ρ^* is such that

$$\gamma(\rho^*) \geqslant 1 + \alpha^*(1-\rho)^2 \tag{19}$$

for some $\alpha^* > 0$. If $\gamma(\rho)$ satisfies (17), and if $\rho = 1 - \varepsilon$, $\rho^* = 1 - \varepsilon^*$, this will be the case, for instance, if

$$\varepsilon^* \geqslant \theta\varepsilon \tag{20}$$

for some fixed $\theta \in (0,1)$. We then have $\alpha^* = \theta^2\alpha$, and the estimate (18) holds in the form

$$\varepsilon_n \leqslant \frac{8}{\alpha\theta^2\rho_1^2}\frac{1}{n}, \qquad n=1,2,\ldots. \tag{21}$$

One practical application of these stability results concerns the experimental fact noted by Grassmann [1979] that in cases where the slit map (Example 5) is applicable it should be used with a parameter $\rho^* > \rho$ in order to avoid corners in the boundary of the image.

8. FAST INITIAL CONVERGENCE

Grassmann's experiments [7] confirm the asymptotically slow rate of convergence predicted by Theorem 6. The same experiments also show that at the beginning the convergence is much faster, especially if augmented osculation families are used. We now present some results aimed at understanding the phenomenon of fast initial convergence.

For any osculation family $\mathcal{F}$, and for $0<\rho<1$, let

$$\psi(\rho) := \inf_{D \in \mathfrak{D}(\rho)} \inf_{h \in \mathcal{F}(D)} \inf_{|z|=\rho} |h(z)|. \tag{22}$$

This is the radius of the largest disk centered at 0 that can always be placed within the image of the circle of radius ρ around 0 under a map $h \in \mathcal{F}(D)$ where $D \in \mathfrak{D}(\rho)$. It follows from Lemma 4 that

$$\psi(\rho) > \rho, \qquad 0<\rho<1. \tag{23}$$

The function ψ can be computed, in principle, for any concretely given osculation family, and it usually turns out that ψ is monotonic.

Example 8. For the Koebe family (Example 1) $\psi(\rho)$ was computed by Julia [13], with the result

$$\begin{aligned}\psi(\rho) &= \sqrt{\rho}\,\frac{1+\rho}{\sqrt{2(1+\rho^2)}+1-\rho}\\ &= \sqrt{\rho}\,\frac{\sqrt{2(1+\rho^2)}-(1-\rho)}{1+\rho}.\end{aligned} \tag{24}$$

The monotonicity of ψ is evident from

$$[\psi(\rho)]^2 = \rho\left\{1 - \frac{2}{\dfrac{\sqrt{2(1+\rho^2)}}{1-\rho}+1}\right\}$$

EXAMPLE 9. For the Ahlfors family (Example 3) it can be verified that

$$\psi(\rho) = \frac{\mathrm{Log}\,2 - \mathrm{Log}(1+\rho^2)}{\mathrm{Log}(1+\rho^{-2}) - \mathrm{Log}\,2} \tag{25}$$

Monotonicity follows by differentiation.

If $D \in \mathfrak{D}(\rho)$ and $h \in \mathfrak{F}(D)$, then by the definition of ψ, $h(D) \in \mathfrak{D}(\rho')$ where $\rho' \geqslant \psi(\rho)$. Thus there follows

THEOREM 7. *If the function ψ defined by* (22) *is monotonic, then the radii ρ_n generated by an osculation algorithm based on $\mathfrak{F}$ satisfy*

$$\rho_n \geqslant \rho_n^*, \qquad n = 1, 2, \ldots,$$

where

$$\rho_1^* := \rho_1, \qquad \rho_{n+1}^* := \psi(\rho_n^*), \quad n = 1, 2, \ldots. \tag{26}$$

In both examples considered above, $\psi(0) = 0$ and $\psi'(0) = \infty$. Thus for small ρ, $\psi(\rho)$ is considerably larger than ρ, and the iteration sequence (26) increases rapidly in the beginning if ρ_1 is sufficiently small. For instance, if $\rho_1^* = 0.1$, only 10 iterations are necessary to achieve $\rho_n^* > 0.4$ if the iteration function (24) is used, and only 7 if the function (25) is used.

What does Theorem 7 yield for large values of n? Does it perhaps yield an improvement of Theorem 6? At least in the two examples considered, this is not the case. In both these examples,

$$\psi(1-\varepsilon) = 1 - \varepsilon + c\varepsilon^3 + O(\varepsilon^4)$$

for some $c > 0$, and for such functions the iteration sequence defined by (26) asymptotically satisfies

$$1 - \rho_n^* \sim \frac{1}{\sqrt{2c}} n^{-1/2}, \qquad n \to \infty$$

(see Polya and Szegö [17, Problem I174]. Thus for large n, the estimate of Theorem 7 is of worse quality than the estimate given by Theorem 6.

The computations required to obtain the explicit formulas for $\psi(\rho)$ given in Examples 8 and 9 require a detailed knowledge of the functions h of the osculation family and thus are fairly tedious. The problem arises to compute, or at least to estimate, the function ψ solely on the basis of the dilatation measure $\gamma(\rho)$ of $\mathfrak{F}$. In the following we provide such an estimate for the more general function

$$\psi(\rho,\sigma) := \inf_{D\in\mathfrak{D}(\rho)} \inf_{h\in\mathfrak{F}(D)} \inf_{|z|=\sigma} |h(z)|, \tag{27}$$

where $0<\rho<1$, $0<\sigma\leqslant\rho$. This function is required for the discussion of osculation algorithms for mapping doubly connected regions.

It is easy to give an upper bound for ψ. Let $h\in\mathfrak{F}(D)$ where $D\in\mathfrak{D}(\rho)$. Because $z^{-1}h(z)$ is analytic and does not vanish for $|z|<\rho$, the minimum of its modulus on every disk $|z|\leqslant\sigma\leqslant\rho$ is assumed on the boundary. The value at O being $h'(0)$, there follows

$$\inf_{|z|=\sigma} \frac{|h(z)|}{\sigma} \leqslant h'(0).$$

Taking the infimum with respect to all $h\in\mathfrak{F}(D)$ and all $D\in\mathfrak{D}(\rho)$, there follows

$$\psi(\rho,\sigma)\leqslant\sigma\gamma(\rho). \tag{28}$$

In particular,

$$\psi(\rho)=\psi(\rho,\rho)\leqslant\rho\gamma(\rho),$$

and by virtue of Theorem 2 there follows

$$\psi(\rho)\leqslant\tfrac{1}{4}(1+\rho)^2.$$

If $\rho=1-\varepsilon$, this means the same as

$$\psi(1-\varepsilon)\leqslant 1-\varepsilon+\tfrac{1}{4}\varepsilon^2, \tag{29}$$

and it follows from the known behavior of an iteration sequence formed with an iteration function satisfying (29) (see Polya and Szegö [17, *loc. cit.*]) that the convergence of an iteration sequence $\{\rho_n^*\}$ formed with ψ at best will satisfy $1 - \rho_n^* = O(1/n)$.

By Lemma 4, $\psi(\rho, \sigma)$ satisfies the trivial lower bound

$$\psi(\rho, \sigma) \geqslant \sigma.$$

To find a less trivial lower bound we consider, for given (ρ, σ) such that $0 < \rho < 1$, $0 < \sigma \leqslant \rho$, the equation for ξ,

$$\xi = \sigma \gamma^{(1-\gamma\xi)/(1+\gamma\xi)}, \tag{30}$$

where $\gamma = \gamma(\rho)$. As a function of ξ, the expression on the right in the interval $[0, \gamma^{-1}]$ strictly decreases from $\sigma\gamma$ to σ. It is thus clear that (30) has a unique solution $\xi \in (0, \gamma^{-1})$. Let this solution be denoted by $\psi_0 = \psi_0(\rho, \sigma)$.

By virtue of $\gamma^{-1} > \rho$, ψ_0 satisfies $\psi_0(\rho, \sigma) > \sigma$ for every $\sigma \leqslant \rho$. Moreover for each fixed ρ, $\psi_0(\rho, \sigma)$ as a function of σ increases from $\psi_0(\rho, 0) = 0$ to

$$\psi_0(\rho) := \psi_0(\rho, \rho).$$

THEOREM 8. *Let $\mathfrak{F}$ be an osculation family with dilatation measure $\gamma(\rho)$. Then for $0 < \rho < 1$, $0 < \sigma \leqslant \rho$,*

$$\psi(\rho, \sigma) \geqslant \psi_0(\rho, \sigma), \tag{31}$$

where ψ_0 is the unique solution in $(0, \gamma^{-1})$ of (30).

Proof. Let $h \in \mathfrak{F}(D)$ where $D \in \mathfrak{D}(\rho)$. Then the function

$$w \to \operatorname{Log}\left|\frac{w}{h^{[-1]}(w)}\right| \tag{32}$$

is harmonic and, by (8), positive at $w = 0$ and at every point $w \in E$ where $h^{[-1]}(w) \neq 0$. We assert that this set includes the disk

$$|w| < \frac{1}{h'(0)}.$$

Indeed, let $h^{[-1]}(w_1)=0$, $w_1 \in E$, $w_1 \neq 0$. Then the modulus of the function

$$w \to \frac{h^{[-1]}(w)}{w} \frac{1-\bar{w}_1 w}{w-w_1}$$

is $\leqslant 1$ on $|w|=1$, and consequently also at O. There follows

$$\frac{1}{h'(0)} \cdot \frac{1}{|w_1|} \leqslant 1,$$

and consequently

$$|w_1| \geqslant \frac{1}{h'(0)}.$$

We thus may apply Harnack's inequality to (32) in the disk $|w| \leqslant \chi^{-1}$, where $\chi := h'(0)$. The result is

$$\operatorname{Log}\left|\frac{w}{h^{[-1]}(w)}\right| \geqslant \frac{1-\chi|w|}{1+\chi|w|} \operatorname{Log} \chi.$$

For any w such that $|h^{[-1]}(w)| = \sigma$ there follows

$$\operatorname{Log} \frac{|w|}{\sigma} \geqslant \frac{1-\chi|w|}{1+\chi|w|} \operatorname{Log} \chi. \tag{33}$$

In particular this holds if w, depending on h, is chosen so that

$$|w| = \inf_{|z|=\sigma} |h(z)|.$$

We now consider a sequence of domains $D_n \in \mathfrak{D}(\rho)$ and functions $h_n \in \mathcal{F}(D_n)$ such that $h_n'(0) \to \gamma(\rho)$. Then the chosen values $w = w_n$ satisfy $|h_n(w_n)| \to \psi := \psi(\rho, \sigma)$, and since (33) holds for each h_n, there follows

$$\operatorname{Log} \frac{\psi}{\sigma} \geqslant \frac{1-\gamma\psi}{1+\gamma\psi} \operatorname{Log} \gamma,$$

or

$$\psi \geqslant \sigma \gamma^{(1-\gamma\psi)/(1+\gamma\psi)}.$$

From the discussion of the equation (30) given above, it is now clear that $\psi \geqslant \psi_0$, as asserted. ■

With Theorem 8 we have achieved our goal to estimate the quantity $\psi(\rho, \sigma)$ in terms of $\gamma(\rho)$ alone. A result analogous to that of Theorem 7 is also available.

COROLLARY 9. *If the function $\psi_0(\rho)$ is monotonic, the radii ρ_n generated by an osculation algorithm based on an osculation family $\mathcal{F}$ with dilatation measure $\gamma(\rho)$ satisfy*

$$\rho_n \geqslant \rho_n', \qquad n = 1, 2, \ldots,$$

where

$$\rho_1' := \rho_1, \qquad \rho_{n+1}' := \psi_0(\rho_n'), \quad n = 1, 2, \ldots.$$

Numerical tests indicate that for the special osculation families where the exact function $\psi(\rho)$ is available, the general estimate of Corollary 9 is only slightly inferior to that of Theorem 7. For instance if $\rho_1' = 0.1$, $\rho_n' > 0.4$ is now achieved in 11 iterations using the dilatation measure for the Koebe family, and in 8 iterations using the dilatation measure for the Ahlfors family.

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